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# Phase Transition in One-Dimensional Lattice Gauge Theories

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Considering one-dimensional nonminimally coupled lattice gauge theories, a class of nonlocal one-dimensional systems is presented which exhibits a phase transition. It is shown that the transition has a latent heat, and therefore is a first-order phase transition.

## INTRODUCTION

During recent decades lattice gauge theories have been extensively studied (Wilson, 1974; Wegner, 1971; Kogut, 1979; Balian *et al.*, 1974, 1975). Lattice theories have no ultraviolet divergences, they provide a nonperturbative approach to some theories, such as QCD (see, e.g., Wilson, 1974), and they are theoretically interesting in themselves. They introduce possibilities which are absent in the continuum; for example, one can consider discrete gauge groups as well as continuous ones. So far, the main interest has been the study of lattice gauge theories (especially pure-gauge theories) on multidimensional lattices (Wegner, 1971; Balian *et al.*, 1975).

The case of one-dimensional lattices, however, is completely different: First, one can consider the general form of (minimally coupled) gauge-invariant interactions, including matter fields as well as gauge fields (Khorrami, 1994). Second, it is a well-known theorem that one-dimensional systems with local interactions do not exhibit phase transition (Domb and Lebowitz, 1986). There are, of course, examples of nonlocal interactions which result in phase transition (Domb and Lebowitz, 1986), but one cannot deduce

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them from general principles. Recently, there has been more interest in one-dimensional systems with phase transition (Mendès France and Tenenbaum, 1993a,b).

Here we discuss in Sections 1 and 2 a class of one-dimensional systems which are natural extensions of minimally coupled gauge-invariant systems (Khorrami, 1994). Then in Section 3 the analytic behavior of the free energy of these systems is considered, and it is shown that, for properly normalized coupling constants, there is a phase transition; in fact, there is a transition temperature above which the pure-gauge interaction of the system is completely eliminated. Below this temperature, the system goes to a minimum-energy state of the pure-gauge interaction, and the effects of nonminimality are lost. Finally, in Section 4 the order of the transition is considered. It is shown that the transition is of first order, and it has a latent heat. The distinguishing features of this class of systems are that, first, they provide an example of one-dimensional systems with phase transition which arise from a general principle (gauge invariance), not an artificial modeling, second, such systems are easily solved, and third, the systems belonging to this class possess a certain kind of universality, that is, the main features of the transition do not depend on the specific system chosen.

## 1. NONMINIMALLY COUPLED LATTICE GAUGE THEORIES

Consider a lattice consisting of a given set of sites $i$ and links $\langle ij \rangle$, two sets $V$ and $\tilde{V}$, a function $\tilde{}: V \to \tilde{V}$, and a multiplication from $\tilde{V} \times V$ to $\tilde{V}$. The Hamiltonian for a nearest neighbor interaction is of the form (Khorrami, 1994; Rebbi).

$$H_0 := -\sum_{\langle ij \rangle} F(\tilde{S}_i S_j) \tag{1.1}$$

where $F$ is a real-valued function and $S$ is the matter field (a $V$-valued function).

Now, suppose that a group $G$ acts on the sets $V$ and $\tilde{V}$ through

$$S \to \hat{g}S \tag{1.2}$$

$$\widetilde{(\hat{g}S)} = \tilde{S}\hat{g}^{-1}$$

where $\hat{g}$ is a representation of $g$. Introducing a group-element-valued field defined on links, one reaches a gauge-invariant Hamiltonian

$$H := H_{\mathrm{m}} + H_G \tag{1.3}$$

where

$$H_{\mathrm{m}} := -\sum_{\langle ij \rangle} F(\tilde{S}_i \hat{U}_{\langle ij \rangle} S_j) \tag{1.4}$$

and

$$H_G := -E(W_{l_1}, W_{l_2}, \ldots) \tag{1.5}$$

In these definitions, $U$ is a $G$-valued field on links, $E$ is a conjugation-invariant (class) real-valued function of its variables, which are members of $G$, and the $W_l$ are Wilson loops of the field $U$ (Khorrami, 1994). It is obvious that the Hamiltonian (1.1) is invariant under global gauge transformation, and (1.3) is invariant under the local gauge transformation

$$\begin{aligned} S_i &\to \hat{g}_i S_i \\ U_{\langle ij \rangle} &\to g_i U_{\langle ij \rangle} g_j^{-1} \end{aligned} \tag{1.6}$$

This is a minimally coupled gauge-invariant Hamiltonian (Khorrami, 1994).

Now, all we need to make $H$ gauge invariant is that $U_{\langle ij \rangle}$ transforms like (1.6). It need not be a group-valued field. If this field (the gauge field) is not group-valued, we have a nonminimal coupling.

## 2. ONE-DIMENSIONAL LATTICE, AND THE GENERAL FORM OF THE PARTITION FUNCTION IN THE THERMODYNAMIC LIMIT

A one-dimensional closed lattice has only one Wilson loop. So the Hamiltonian (1.3) takes the form

$$H = -\sum_{i=1/2}^{N-1/2} F(\tilde{S}_{i-1/2} \hat{U}_i S_{i+1/2}) - E\left(\prod_{i=1/2}^{N-1/2} U_i\right) \tag{2.1}$$

where $N$ is the number of lattice sites,

$$X_{N+k} := X_k \tag{2.2}$$

and (Khorrami, 1994)

$$U_i := U_{\langle i-1/2, i+1/2 \rangle} \tag{2.3}$$

Our main goal is to calculate the partition function

$$Z := \int \left(\prod_i dS_i\right)\left(\prod_j dU_j\right) \exp[-\beta H(\mathbf{S}, \mathbf{U})] \tag{2.4}$$

where boldface quantities refer to the set of corresponding quantities on every site (or link). We also assume that the integration measures are invariant under the action of group. So, defining a partial partition function

$$Z_{\mathrm{m}} := \int D\mathbf{S} \prod_i f(\tilde{S}_{i-1/2}\hat{U}_i S_{i+1/2}) \tag{2.5}$$

where

$$\begin{cases} f := \exp(\beta F) \\ e := \exp(\beta E) \end{cases} \tag{2.6}$$

it is easy to see that

$$Z_{\mathrm{m}} = \int D\mathbf{S}\, D\mathbf{g} \prod_i f(\tilde{S}_{i-1/2}\hat{g}_{i-1/2}\hat{U}_i\hat{g}_{i+1/2}S_{i+1/2}) \tag{2.7}$$

where we have normalized the group volume to one.

Now, defining a linear operator $P(U, S, S')$ on the functionals of $G$ through

$$(\psi P)(g) := \int dg'\ \psi(g') f(\tilde{S}' g'^{-1}\hat{U}\hat{g}S) \tag{2.8}$$

one can see that

$$Z_{\mathrm{m}} = \int D\mathbf{S}\ \mathrm{tr}\left[\prod_i P(U_i, S_{i-1/2}, S_{i+1/2})\right] \tag{2.9}$$

We want to prove that the eigenvector of $P$ corresponding to its largest eigenvalue is independent of its arguments, and that the largest eigenvalue itself depends only on the orbits of the arguments of $P$, provided that $G$ is compact and the equation

$$Ug = g''U \tag{2.10}$$

always has a solution for $g''$.

To establish these properties, we observe that

$$(\psi P)(g) \leq \psi(g_{\max}) \int dg'\ f(\tilde{S}' g'^{-1}\hat{U}\hat{g}S) \tag{2.11}$$

where $g_{\max}$ is the element of the group on which $\psi$ attains its maximum value. This point exists, since the group is compact. Using the existence of a solution for (2.10) and the invariance of the group measure under group translations, one can write the above inequality as

$$(\psi P)(g) \leq \psi(g_{\max}) \int dg' \, f(\tilde{S}' \hat{g}'^{-1} \hat{U} S) \tag{2.12}$$

This inequality also holds when $\psi$ is an eigenvector. So,

$$\lambda \psi(g) \leq \psi(g_{\max}) \int dg' \, f(\tilde{S}' \hat{g}'^{-1} \hat{U} S) \tag{2.13}$$

where $\lambda$ is the corresponding eigenvalue. In the special case $g = g_{\max}$, one has

$$\lambda \psi(g_{\max}) \leq \psi(g_{\max}) \int dg' \, f(\tilde{S}' \hat{g}'^{-1} \hat{U} S) \tag{2.14}$$

One can always make $\psi(g_{\max})$ positive. This implies that

$$\lambda \leq \int dg' \, f(\tilde{S}' \hat{g}'^{-1} \hat{U} S) \tag{2.15}$$

It is also seen that the right-hand side of (2.15) is attained for the constant function. So the largest eigenvalue of $P$ is

$$\mu(U, S, S') := \int dg' \, f(\tilde{S}' \hat{g}'^{-1} \hat{U} S) \tag{2.16}$$

Using the existence of a solution for (2.10) and the invariance of the group measure under group translations, one can also see that

$$\mu(gUg'^{-1}, \hat{g}'' S, \hat{g}''' S') = \mu(U, S, S') \tag{2.17}$$

which is what we wanted to prove.

So, in the thermodynamic limit we have

$$Z = \int D\mathbf{S} \, D\mathbf{U} \, e\left(\prod_i U_i\right)\left[\prod_j \mu(|U_j|, |S_{j+1/2}|, |S'_{j-1/2}|)\right] \tag{2.18}$$

where the *absolute value* means the orbit of the element under the action of $G$.

Now, we have

$$\begin{aligned} &\int D\mathbf{U} \, e\left(\prod_i U_i\right)\left[\prod_j \mu(|U_j|, |S_{j+1/2}|, |S'_{j-1/2}|)\right] \\ &= \int D\mathbf{U} \, e\left(g \prod_i U_i\right)\left[\prod_j \mu(|U_j|, |S_{j+1/2}|, |S'_{j-1/2}|)\right] \\ &= \int D\mathbf{U} \, dg \, e\left(g \prod_i U_i\right)\left[\prod_j \mu(|U_j|, |S_{j+1/2}|, |S'_{j-1/2}|)\right] \end{aligned} \tag{2.19}$$

Using the existence of a solution in (2.10), it is easy to show that the integral

$$\nu(\mathbf{U}) := \int dg\, e\left(g \prod_i U_i\right) \tag{2.20}$$

depends only on the orbits of the $U_i$; in fact, it depends on the product of the orbits:

$$\nu(\mathbf{U}) = \nu\left(\prod_i |U_i|\right) \tag{2.21}$$

The product on the right-hand side of (2.21) is well defined, since we have

$$g_1 U_1 g_1'^{-1} g_2 U_2 g_2'^{-1} = g_1 g' U_1 U_2 g_2'^{-1} \tag{2.22}$$

So one can define the product of two orbits as the orbit of the product of two arbitrary elements, one from each orbit.

To conclude, one can write the partition function as

$$Z = \int D\mathbf{S}\, D\mathbf{U}\, \nu\left(\prod_i |U_i|\right)\left[\prod_j \mu(|U_j|, |S_{j+1/2}|, |S'_{j-1/2}|)\right] \tag{2.23}$$

This relation holds, provided that the group $G$ is compact, that equation (2.10) has always a solution for $g''$, and that the integration measures are invariant under the action of the group.

## 3. A CLASS OF ONE-DIMENSIONAL SYSTEMS WITH PHASE TRANSITION

Suppose that the matter-field space consists of a single orbit of the gauge group. It is then easy to show that the partial partition function

$$Z_G := \int D\mathbf{U} \exp[-\beta H(\mathbf{S}, \mathbf{U})] \tag{3.1}$$

does not depend on $\mathbf{S}$ (Khorrami, 1994). This means that one can eliminate the matter field from the Hamiltonian and use a gauge-fixed Hamiltonian

$$H_{\mathrm{gf}} := -\sum_i F^0(U_i) - E\left(\prod_i U_i\right) \tag{3.2}$$

where

$$F^0(U) := F(\tilde{S}^0 \hat{U} S^0) \tag{3.3}$$

and $S^0$ is an arbitrary member of the matter-field space. In this case one has

$$Z = \left(\int dS\right)^N \int D\mathbf{U} \exp(-\beta H_{\mathrm{gf}}) \tag{3.4}$$

This result holds even for finite lattices. In the thermodynamic limit, using (2.23), we have

$$Z = \left(\int dS\right)^N \int D\mathbf{U}\, \nu\left(\prod_i |U_i|\right)\left[\prod_j \mu(|U_j|)\right] \tag{3.5}$$

Now, take a special form for the gauge field: the formal product of a real number in a set $\{a_m\}$ and a member of the gauge group:

$$U = vW \tag{3.6}$$

where

$$W \in G \tag{3.7}$$

and

$$v \in \{a_m\} \tag{3.8}$$

We also assume that the functions $E$ and $F$ are linear with respect to $v$'s. So we have

$$H_{\mathrm{gf}} = -\sum_j v_j F^0(W_j) - \left(\prod_j v_j\right) E\left(\prod_j W_j\right) \tag{3.9}$$

$E$ is a bounded function, and its bound does not depend on $N$. Therefore the maximum of $|a_m|$ should be 1, so that neither does $\ln Z$ per site diverge in the thermodynamic limit nor does the pure-gauge part of the interaction disappear in this limit. We now rewrite (3.9) as

$$H_{\mathrm{gf}} = -J \sum_j v_j F^0(W_j) - K\left(\prod_j v_j\right) E\left(\prod_i W_j\right) \tag{3.10}$$

where we have assumed that the maxima of $F^0$ and $E$ in (3.10) are one, and $\{a_m\}$ is a subset of $[0, 1]$. One can then rewrite (3.5) as

$$Z_{\mathrm{gf}} = \sum_{\{v_j\}} \nu\left(\prod_i v_i\right)\left[\prod_j \mu(v_j)\right] \tag{3.11}$$

or

$$Z_{\mathrm{gf}} = \sum_{\{v_j\}} I_0^{(E)}\left[\beta K\left(\prod_i v_i\right)\right]\left[\prod_j I_0^{(F^0)}(\beta J v_j)\right] \tag{3.12}$$

where we have defined

$$I_0^{(F)}(x) := \int dg \, \exp[xF(g)] \tag{3.13}$$

Writing the Taylor series for $I^{(E)}(x)$,

$$I^{(E)}(x) = \sum_{n=0}^{\infty} \alpha_{0n}^{(E)} x^n \tag{3.14}$$

we will have

$$\begin{aligned} Z_{\mathrm{gf}} &= \sum_{\{v_j\}} \sum_{n=0}^{\infty} (\beta K)^n \alpha_{0n}^{(E)} \prod_j [v_j^n I_0^{(F^0)}(\beta J v_j)] \\ &= \sum_{n=0}^{\infty} (\beta K)^n \alpha_{0n}^{(E)} \left\{ \sum_m [a_m^n I_0^{(F^0)}(\beta J a_m)] \right\}^N \end{aligned} \tag{3.15}$$

The ratio of different terms of this series varies exponentially with $N$. So in the thermodynamic limit only the largest term contributes. We have then (if $\{a_m\} \neq \{1\}$)

$$\begin{aligned} Z_{\mathrm{gf}} &= \alpha_{00}^{(E)} \left\{ \sum_m [I_0^{(F^0)}(\beta J a_m)] \right\}^N \\ &= \alpha_{00}^{(E)} \left\{ \sum_m [I_0^{(F^0)}(\beta J a_m)] \right\}^N \end{aligned} \tag{3.16}$$

But in this partition function there is no trace of the pure-gauge interaction. One can restore this interaction through renormalizing the coupling constant:

$$K =: \kappa N^x \tag{3.17}$$

We take $\kappa$ to be constant and

$$0 < x \leq 1 \tag{3.18}$$

To make ln $Z$ per site finite, $x$ should not be greater than 1. One then has

$$\begin{aligned} Z_{\mathrm{gf}} &= \sum_{n=0}^{\infty} (\beta \kappa N^x)^n \alpha_{0n}^{(E)} \left\{ \sum_m [a_m^n I_0^{(F^0)}(\beta J a_m)] \right\}^N \\ &=: \sum_n Z_n \end{aligned} \tag{3.19}$$

Now, there is a local maximum for $Z_n$ aside from $Z_0$. Assuming $n_{\max}$ to be large, and using the fact that $I_0^{(E)}(x)$ behaves like $\exp(x)$ for large $x$, we have

$$\alpha_{0n}^{(E)} \sim \frac{1}{n!} \qquad \text{for} \quad n \text{ large} \tag{3.20}$$

$$\ln Z_n \sim n - \ln n + n \ln(\beta\kappa N^x) + N \ln[I_0^{(F0)}(\beta J)] \qquad \text{for} \quad n \text{ large} \tag{3.21}$$

which yields

$$\frac{d \ln Z_n}{dn} = \ln \frac{\beta\kappa N^x}{n} \tag{3.22}$$

or

$$n_{\max} = \beta\kappa N^x \tag{3.23}$$

But

$$\ln Z_{n_{\max}} = \beta\kappa N^x + N \ln[I_0^{(F0)}(\beta J)] \tag{3.24}$$

is a local maximum. This should be compared with $\ln Z_0$, which is another local maximum:

$$\ln Z_0 = N \ln\left[\sum_m I_0^{(F0)}(\beta J a_m)\right] \tag{3.25}$$

The greater term determines the partition function. But we have

$$\ln \frac{Z_0}{Z_{n_{\max}}} = N\left[\ln \frac{\sum_m I_0^{(F0)}(\beta J a_m)}{I_0^{(F0)}(\beta J)} - \beta\kappa N^{x-1}\right] \tag{3.26}$$

If $x < 1$, this expression is always positive for large $N$, which means that the pure-gauge interaction is eliminated. However, if $x = 1$, there is a particular value for $\beta$, $\beta_t$, at which this expression changes sign. So we have

$$\frac{1}{N} \ln Z = \begin{cases} \ln\left[\sum_m I_0^{(F0)}(\beta J a_m)\right], & \beta < \beta_t \quad (T > T_t) \\ \ln[I_0^{(F0)}(\beta J)] + \beta\kappa, & \beta > \beta_t \quad (T < T_t) \end{cases} \tag{3.27}$$

It is seen that above $T_t$ the partition function is independent of $\kappa$, that is, the system becomes independent of the pure-gauge interaction. Below $T_t$ the system is independent of the $a_m$, that is, the system is frozen in $a_m = 1$ and a value for the $W_i$ for which the function $E(\Pi_i W_i)$ is maximum, 1. So, for $T > T_t$ the system does not see the pure-gauge interaction, whereas for $T < T_t$ the system goes to the state of minimum energy (of the pure-gauge interaction).

Also note that this renormalization of the coupling constant has a simple meaning; it means that the pure-gauge interaction introduced in (3.10), is in fact an interaction density, but a density which is uniform on the lattice.

## 4. ORDER OF THE TRANSITION

From (3.27), we have

$$\frac{\partial}{\partial\beta}\left(\frac{1}{N}\ln Z\right)=\begin{cases} J\dfrac{\sum\limits_m I_0'^{(F^0)}(\beta J a_m)}{\sum\limits_m I_0^{(F^0)}(\beta J a_m)}, & \beta<\beta_t \quad (T>T_t) \\ J\dfrac{I_0'^{(F^0)}(\beta J)}{I_0^{(F^0)}(\beta J)}+\kappa, & \beta>\beta_t \quad (T<T_t) \end{cases} \tag{4.1}$$

Now,

$$\begin{aligned} \Delta S &= -\Delta\left(\frac{\partial A}{\partial T}\right) \\ &= \frac{\beta}{T}\Delta\left(\frac{\partial A}{\partial\beta}\right) \\ &= -\frac{1}{T}\Delta\left(\frac{\partial \ln Z}{\partial\beta}\right) \\ &= \frac{1}{T}\left\{J\left[\frac{I_0'^{(F^0)}(\beta J)}{I_0^{(F^0)}(\beta J)}-\frac{\sum\limits_m a_m I_0'^{(F^0)}(\beta J a_m)}{\sum\limits_m I_0'^{(F^0)}(\beta J a_m)}\right]+\kappa\right\} \end{aligned} \tag{4.2}$$

where $S$ and $A$ are the entropy and the free energy of the system, respectively. We will see that a sufficient condition for $\Delta S$ to be positive is that

$$\frac{d}{dx}\left[\frac{dI_0^{(F^0)}(x)/dx}{I_0^{(F^0)}(x)}\right]>0 \tag{4.3}$$

But, for any system we have

$$C_V=\frac{\beta}{T}\frac{\partial^2 Z}{\partial\beta^2}>0 \tag{4.4}$$

Using this for a system with $\kappa=0$ and $\{a_m\}=\{1\}$, one can prove (4.3). It is now easy to prove that $\Delta S>0$:

$$\Delta S = \frac{1}{T}\left\{ J \frac{I_0'^{(F^0)}(\beta J)}{I_0^{(F^0)}(\beta J)} \left[ 1 - \frac{\sum_m I_0'^{(F^0)}(\beta J a_m)/I_0'^{(F^0)}(\beta J)}{\sum_m I_0^{(F^0)}(\beta J a_m)/I_0^{(F^0)}(\beta J)} \right] + \kappa \right\} \tag{4.5}$$

From (4.3), we have

$$\frac{I_0'^{(F^0)}(\beta J a_m)}{I_0^{(F^0)}(\beta J a_m)} < \frac{I_0'^{(F^0)}(\beta J)}{I_0^{(F^0)}(\beta J)} \tag{4.6}$$

or

$$\frac{I_0'^{(F^0)}(\beta J a_m)}{I_0'^{(F^0)}(\beta J)} < \frac{I_0^{(F^0)}(\beta J a_m)}{I_0^{(F^0)}(\beta J)} \tag{4.7}$$

Inserting this inequality in (4.5), one concludes that

$$\Delta S > 0 \tag{4.8}$$

This means that the phase transition has a latent heat. So it is a first-order phase transition.

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